Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(01)00010-7

J. Appl. Maths Mechs, Vol. 65, No. 1, pp. 89–108, 2001

0021-8928/01/\$—see front matter

STABLE SHOCK WAVES IN TWO-LAYER "SHALLOW WATER"†

V. V. OSTAPENKO

Novosibirsk

(*Received* 17 *May* 2000)

A criterion is proposed for the correctness of a complete system of conservation laws which assumes the maximum compatibility between the domain of convexity of a closing conservation law and the domain of hyperbolicity of the differential model. A correct complete system of conservation laws is chosen on the basis of this criterion for a model of two-layer "shallow water" with a free upper boundary in which the laws of conservation of mass in the layers, of total momentum and discontinuity in the velocity at the interface of the layers are the basic conservation laws and the law of conservation of the total energy is the closing conservation law. An analysis of the stable shock (intermittent) waves permitted by this system is presented. © 2001 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Three differential models of two-layer "shallow water" were derived and analysed in [1]: model I is a model with a free upper boundary, II is a model "under a cover" and their common limit case is model III. Hyperbolicity domains were constructed for these models and, in this connection, the problem arose of formulating these models in the form of complete system of conservation laws and of investigating the stable discontinuous solutions with shock waves which are permitted by these systems.

One of the possible approaches to solving this problem, which dates back to the classical model of one-dimensional gas dynamics [2], is associated with the proof of the unique solvability of the problem of the decay of an arbitrary discontinuity in the hyperbolicity domain. This approach has been implemented in the case of the simplest model III [3]. However, it was impossible to extend this successfully to the more complex models I and II since the problems which arise here concerning the decay of a discontinuity are too complex, and it has been found to be impossible to carry out a complete mathematical analysis of them in the general case.

Subsequently, more structured models were developed for investigating the wave flows of stratified "shallow water" in which the interface between the two principal layers is replaced by a special mixing layer [4] or a continuous change in the density along the vertical is introduced [5, 6]. However, the basis of these models will not be sufficiently useful without carrying out a comparative analysis of them with the complete systems of conservation laws for the basic models of two-layer "shallow water" (models I and II), which are their simple limiting cases.

In this connection, a solution of the problem formulated in [1] is proposed here for model I with a free upper boundary, which is of the greatest physical interest. This solution was obtained using the criterion of the complete system of conservation laws presented below, which assumes that there is maximum compatibility between the domain of convexity of the closing conservation law and the domain of hyperbolicity of the differential model.

2. THE CRITERION FOR THE CORRECTNESS OF THE COMPLETE SYSTEM OF CONSERVATION LAWS

The hyperbolic system of conservation laws [2, 7]

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \tag{2.1}$$

$$\mathbf{u}(t, x) = (u_1, \ldots, u_m), \quad \mathbf{f}(\mathbf{u}) = (f_1, \ldots, f_m)$$

is called the complete system [8] with a convex extension [9] if it allows of the additional closing conservation law

†*Prikl. Mat. Mekh.* Vol. 65, No. 1, pp. 94–113, 2001.

$$U(\mathbf{u})_t + F(\mathbf{u})_x = 0 \tag{2.2}$$

where $U_\mathbf{u} \cdot f_\mathbf{u} = F_\mathbf{u}$ with a convex function (entropy function) $U(\mathbf{u})$, that is, $U_{\mathbf{uu}} > 0$. It was shown in [8–11] that such a system in the canonical variables $\mathbf{w} = \mathbf{w}(\mathbf{u}) = U_\mathbf{u}$ can be written in the symmetric form

$$A\mathbf{w}_t + B\mathbf{w}_x = 0$$

$$A = \Phi_{\mathbf{ww}} > 0, \quad \Phi = \mathbf{w} \cdot \mathbf{u} - U; \quad B = \Psi_{\mathbf{ww}}, \quad \Psi = \mathbf{w} \cdot \mathbf{f} - F$$

The closing conservation law (2.2) of the complete system (2.1) was proposed for use [7, 9] in selecting the stable discontinuous solutions, as solutions of (2.1), which, in a weak sense [7], satisfy the entropy inequality

$$U(\mathbf{u})_t + F(\mathbf{u})_x \leqslant 0 \tag{2.3}$$

In the shock wave front $x = x(t)$, this inequality takes the form

$$[DU - F] < 0 \tag{2.4}$$

where $D = x'(t)$ is the propagation velocity of the shock wave and $[f]$ is the discontinuity in the function $f(t, x)$ on the shock wave front, that is

$$[f] = f_1 - f_0, \quad f_1 = f(t, x(t) - 0), \quad f_0 = f(t, x(t) + 0)$$

It has been suggested in many papers (see [12–14], for example) that, in the case of complete system of conservation laws (2.1) with a convex extension (2.2), the entropy condition of stability (2.3), (2.4) guarantees the unique solvability "in the large" of the Cauchy problem in a certain class of piecewise-continuous functions. In this connection, the existence of the convex extension (2.2) was put forward as the decisive requirement in the correct formulation of the hyperbolic system in the form of a complete system of conservation laws. However, the rigorous application of this requirement assumes that there is a complete match between the domain of convexity of the closing conservation law and the domain of hyperbolicity of the differential model, which many actual models cannot satisfy in principle (in particular, not one of the models of two-layer "shallow water" proposed in [1] meets this requirement).

As a result, at the present time, in the theory of hyperbolic systems, a quite unusual situation has built up when some researchers at once reject whole classes of differential models which do not permit a convex extension in the whole of the hyperbolicity domain as being incorrect while others who are working with these models completely ignore the very idea of a convex extension, in spite of the fact that is arises in the most natural way in the most developed hyperbolic systems such as, for example, the system of conservation laws in gas dynamics.

A compromise approach is proposed in this paper which, on the one hand, reconciles these two opposing points of view and, on the other hand, allows of a far wider use of the concept of a convex extension to analyse the correctness of a complete system of conservation laws. Within the framework of this "softer" approach, the domain of convexity Ω^c of the entropy function $U(u)$ of the correct complete system may or may not coincide with the whole of its hyperbolic domain Ω^h (here, $\Omega^c \subseteq \Omega^h$ always), but it must have the maximum compatibility with it when compared with all the other possible complete systems which are permitted by the differential model. In particular, if we have two different complete systems of conservation laws with domains of convexity Ω_1^c and Ω_2^c which are obtained from one and the same differential hyperbolic system (2.1) and, at the same time, $\Omega_2^c \subset \Omega_1^c$, the complete system with the domain of convexity Ω_1^c will satisfy the given criterion of correctness to a greater degree.

In Section 3, this criterion for the correctness of a complete system of conservation laws is tested using the example of the well-known model of single-layer shallow water and, in Section 4, it is applied to select the correct complete system of conservation laws in a model of two-layer "shallow water' with a free upper boundary. The shock waves which are permitted by the Hugoniot conditions of this complete system are studied in Section 5 and those which are permitted by the condition for the convexity of its entropy function (total energy) are investigated in Section 6. In Section 7, those shockwaves which are stable are picked out using the entropy (energy) inequality (2.4). Special two-layer discontinuous waves in which there is no flow across a discontinuity in one of the layers (such discontinuous waves are called "natural dams") are investigated in Section 8. These results are extended to the spatially two-dimensional case in Sections 9–12.

3. SINGLE-LAYER "SHALLOW WATER"

It is well known [2, 8] that, in the complete system of conservation laws of single-layer "shallow water", the laws of conservation of mass

$$h_t + q_x = 0 \tag{3.1}$$

and of total momentum

$$q_t + (qv + gh^2/2)_x = 0 \tag{3.2}$$

appear as the basic conservation laws while the law of conservation of total energy

$$e_t + (q(v^2 + 2gh))_x = 0 \tag{3.3}$$

appears as the closing conservation law where h is the depth of the flow, q is the flow rate, $v = q/h$ is the velocity, g is the acceleration due to gravity and $e = qv + gh^2$ is the total energy, which, in the given case, plays the role of the entropy function. On writing the total energy as a function of the basis variables $u = (h, q)$, we obtain

$$e(h, q) = q^2/h + gh^2 \tag{3.4}$$

Since this function is convex when $h > 0$, that is, in the whole of the hyperbolic domain of system (3.1), (3.2), the domain of convexity of the "entropy function" (of the total energy (3.4)) of the complete system (3.1)–(3.3) is thereby ideally matched with its hyperbolicity domain (these domains are identical).

The equation

$$v_t + (v^2/2 + gh)_x = 0 \tag{3.5}$$

is the differential corollary of system (3.1) and (3.2) and is the law of conservation of local momentum of each fluid particle along its streamline. Unlike in the case of the complete system (3.1)–(3.3), the complete system which is obtained with the basic conservation laws (3.1) and (3.5) and the same closing law (3.3) has an "entropy function" (total energy).

$$e(h, v) = v^2h + gh^2 \tag{3.6}$$

which is only convex in the case of subcritical flows $|v| < \sqrt{gh}$ and, hence, system (3.1)–(3.3) satisfies the criterion of correctness, which has been proposed above to a greater degree than system (3.1), (3.3), (3.5).

Finally, if one takes the laws of conservation of mass (3.1) and local momentum (3.5) as the basic conservation laws and the law of conservation of total momentum (3.2) as the closing conservation law, then the "entropy function" (the total momentum) $q(h, v) = h \cdot v$ of such a "totally unphysical" complete system is not convex for all values of the basic variables h and v. Hence, in the case of the well studied model of single-layer 'shallow water", the criterion proposed above works quite successfully by uniquely picking out the physically correct complete system (3.1)–(3.3).

We now recall how the stable shock (discontinuous) waves in system (3.1)–(3.3) are picked out using the the entropy (2.4) (energy) condition. For this, changing to a system of coordinates in which the velocity of the shock wave $D = 0$ and writing in it the Hugoniot conditions for the basic conservation laws (3.1) and (3.2) and the entropy condition of stability (2.4), we obtain for the closing total energy conservation laws (3.3)

$$[q] = 0 \Rightarrow q_1 = q_0 = q \neq 0 \tag{3.7}$$

$$[qv + gh^2/2] = q^2[h^{-1}] + g[h^2]/2 = 0 \tag{3.8}$$

$$[q(v^2 + 2gh)] = q[v^2 + 2gh] > 0 \tag{3.9}$$

Assuming, to fix our ideas, that $q > 0$, that is, that the flow across the discontinuity is directed along the x axis, we rewrite the energy inequality (3.9) in the form

$$[v^2 + 2gh] = [h](2g - q^2(h_1 + h_0)/(h_1h_0)^2) > 0 \tag{3.10}$$

Since, as follows from (3.8)

$$q^2 = -g[h^2]/(2[h^{-1}]) = gh_1h_0(h_1 + h_0)/2 \tag{3.11}$$

on substituting this value of the flow rate into inequality (3.1), we finally have

$$g[h](4-(h_1+h_0)^2/(2h_1h_0)) = -g[h]^3/(2h_1h_0) > 0$$

This means that, when $q > 0$, for a stable shock wave

$$[h] = h_1 - h_0 < 0 \Rightarrow h_1 < h_0$$

that is, the depth of the water increases when it moves across the shock wave front.

It is interesting to note that the Hugoniot conditions, written when $D = 0$ for the law of conservation of local momentum (3.5), have the form

$$[v^2/2 + gh] = 0$$

It follows from this that, if the laws of conservation of mass (3.1) and local momentum (3.5) are taken as the basic laws, then, by virtue of (3.7) and (3.9), the total energy in this system at the discontinuities will also be conserved, as a consequence of which conservation law (3.3) corresponding to it cannot be used as the closing conservation law to select the stable shock waves.

The energy method described here will next be applied to pick out the stable shock waves in two-layer "shallow water".

4. TWO-LAYER "SHALLOW WATER"

The differential equations for two-layer "shallow water" with a free upper boundary have the form [1]

$$h_t + q_x = 0, \quad H_t + Q_x = 0 \tag{4.1}$$

$$v_t + (v^2/2 + g(h+H))_x = 0, \quad V_t + (V^2/2 + g(H + \lambda h))_x = 0 \tag{4.2}$$

where h, q, $v = q/h$ are the depth, the flow rate and the velocity in the upper layer, H, Q, $V = Q/H$ are the same quantities in the lower layer and $\lambda < 1$ is the ratio of the densities of the upper and lower layers. Equations (4.1) are the laws of conservation of mass and Eqs (4.2) are the laws of conservation of local momentum in each of the fluid layers.

It has been shown [1] that system (4.1), (4.2) has two further, linearly independent conservation laws: of the total momentum

$$\alpha = Q + \lambda q \tag{4.3}$$

in the form

$$\alpha_t + (QV + \lambda qv + g\varphi/2)_x = 0; \quad \varphi = H^2 + \lambda h^2 + 2\lambda Hh \tag{4.4}$$

and of the total energy

$$e = QV + \lambda qv + g\varphi \tag{4.5}$$

in the form

$$e_t + (Q(V^2 + 2gH_m) + \lambda q(v^2 + 2gH_n))_x = 0, \tag{4.6}$$
$$H_m = H + \lambda h, \quad H_n = H + h$$

By analogy with the single-layer case, we take the laws of conservation of mass in the layers (4.1) and the law of conservation of the total momentum (4.4) as the basic laws and the law of conservation of the total energy (4.6) as the closing law. As a result, the classical problem arises of the choice of the one further missing (in this case, the fourth) conservation law.

It has been shown [15] that the use of the criterion proposed here uniquely singles out the law of conservation of the velocity jump $\gamma = V - v$ on the interface of the layers, following from (4.2), as this fourth basic conservation law

$$\gamma_t + ((V^2 - v^2)/2 - \mu g h)_x = 0; \quad \mu = 1 - \lambda \tag{4.7}$$

This is associated with the fact that the total energy (4.5), written as the function

$$e(h, H, \alpha, \gamma) = (\alpha^2 + \lambda h H \gamma^2)/(H + \lambda h) + g(H^2 + \lambda h^2 + 2\lambda h H)$$

which depends on the basic variables $\mathbf{u} = (h, H, \alpha, \gamma)$ is convex subject to the condition

$$|\bar{\gamma}| < \frac{b}{\lambda r}\left(\frac{g}{2}(c - (c^2 - 4\lambda^2 \mu r^3)^{1/2})\right)^{1/2}, \quad b = 1 + \lambda r, \quad c = 1 + \lambda^2 r^3 \tag{4.8}$$

where

$$\bar{\gamma} = \gamma/\sqrt{H} = (V - v)/\sqrt{H}, \quad r = h/H \tag{4.9}$$

are the same variables in which the condition for the hyperbolic form of system (4.1), (4.2), which was obtained in an implicit form in [1], is written. If, however, instead of (4.7), any other conservation law having a physical meaning is taken as the fourth basic conservation law such as, for example, the law of conservation of local momentum in one of the layers (4.2) or the law of conservation of the jump in the local momentum $\beta = V - \lambda v$ on the interface of the layers or, finally, the laws of conservation of mass (4.1) and of local momentum (4.2) in the layers, are taken as the basis laws then, in all of these cases, the domain of convexity of the total energy (4.5) as a function of the corresponding basis variables is a constraint on the quantities, which are different from the variables in (4.9), in which the condition for system (4.1), (4.2) to be hyperbolic is written.

That, for example, if the laws of conservation of mass (4.1) and local momentum (4.2) are chosen as the basis in the layers (such a complete system is analogous to the single-layer system (3.1), (3.3), (3.5)), then the total energy (4.5) as a function

$$e(h, H, v, V) = HV^2 + \lambda h v^2 + g(H^2 + \lambda h^2 + 2\lambda h H)$$

of the basis variables $u = (h, H, v, V)$ is convex subject to the condition of the sub-criticality of the flows in the two layers

$$|v| < \sqrt{gh}, \quad |V| < \sqrt{gH} \tag{4.10}$$

and subject to the additional condition

$$(gh - v^2)(gH - V^2) > \lambda g^2 h H$$

which reinforces (4.10).

It is obvious that these conditions, which require the boundedness of the velocity in each fluid layer, are not compatible with the condition that system (4.1), (4.2) is hyperbolic, which only assumes that there is a constraint on the difference between the velocities in the layers.

Note that the additional basis law of conservation (4.7) also arises in limit cases of models which have previously been proposed [5, 6].

5. SHOCK WAVES PERMITTED BY THE HUGONIOT CONDITIONS

Writing the Hugoniot conditions for system (4.1), (4.4), (4.7) in a system of coordinates in which the shock wave velocity $D = 0$, we obtain the following relations which are a two-layer analogue of conditions (3.7) and (3.8)

$$[q] = 0 \Rightarrow q_1 = q_0 = q, \quad [Q] = 0 \Rightarrow Q_1 = Q_0 = Q \tag{5.1}$$

$$[QV + \lambda q v + g\varphi/2] = Q^2[H^{-1}] + \lambda q^2[h^{-1}] + g[\varphi]/2 = 0 \tag{5.2}$$

$$[V^2 - v^2 - 2\mu g h] = Q^2[H^{-2}] - q^2[h^{-2}] - 2\mu g[h] = 0 \tag{5.3}$$

Equations (5.2) and (5.3) are a linear system in Q^2 and q^2. Solving this system, taking account of the fact that $[h] \neq 0$ and $[H] \neq 0$ (otherwise, no discontinuity exists), we find

$$Q^2 = \frac{g(4\lambda\mu[h] - \tilde{h}[\varphi])}{2\tilde{b}[H^{-1}]}, \quad q^2 = -\frac{g(4\mu[h] + \tilde{H}[\varphi])}{2\tilde{b}[h^{-1}]} \tag{5.4}$$

$$\tilde{b} = \tilde{h} + \lambda\tilde{H}, \quad \tilde{h} = h_1^{-1} + h_0^{-1}, \quad \tilde{H} = H_1^{-1} + H_0^{-1}$$

The non-negative property of the right-hand sides in (5.4) (which are the two-layer analogue of relation (3.11)) leads to the following constraints on the change in depth at the discontinuity

$$(\tilde{h}[\varphi] - 4\lambda\mu[h])[H] \geqslant 0, \quad (\tilde{H}[\varphi] + 4\mu[h])[h] \geqslant 0 \tag{5.5}$$

Further, without loss of generality, we shall assume that

$$H_0 = 1, \quad [H] < 0 \Rightarrow H_1 < H_0 = 1 \tag{5.6}$$

If follows from this that the first of the inequalities in (5.5) can be rewritten in the form

$$4\lambda\mu[h] \geqslant \tilde{h}[\varphi] \tag{5.7}$$

If the condition

$$[h] < 0 \Rightarrow h_1 < h_0 \tag{5.8}$$

holds simultaneously with (5.6), then the fact that the second inequality of (5.5) is satisfied follows at once from the relation

$$[\varphi] = [H^2 + \lambda h^2 + 2\lambda H h] < 0$$

while the fact that inequality (5.7) is satisfied follows from the chain of inequalities

$$4\lambda\mu[h] - \tilde{h}[\varphi] > 4\lambda\mu[h] - \lambda\tilde{h}[h^2] > \lambda[h](4 - \tilde{h}\bar{h}) =$$

$$= -\lambda[h]^3/(h_1 h_0) > 0, \quad \bar{h} = h_1 + h_0$$

This means that any discontinuities for which the depth in the two layers simultaneously decreases or increases upon passing through the shock were front are permitted by conditions (5.5).

The set of permissible depths (5.8) is located within the angle $A_0 h_0$ in Fig. 1(a) in the plane of the variables (h_0, h_1).

We will now consider the case when

$$[h] > 0 \Rightarrow h_1 > h_0 \tag{5.9}$$

when the second inequality of (5.5) can be written in the form

$$\tilde{H}[\varphi] + 4\mu[h] \geqslant 0 \tag{5.10}$$

Applying formula $[\varphi] = \lambda[H_n^2] + \mu[H^2]$, we transform inequality (5.1) in the following manner

$$\tilde{H}(\lambda[H_n^2] + \mu[H^2]) + 4\mu[H_n - H] = (\lambda\tilde{H}\bar{H}_n + 4\mu)[H_n] + \mu(\tilde{H}\bar{H} - 4)[H] =$$

$$= (a[H_n] + \mu[H]^3)/(H_1 H_0) \geqslant 0$$

$$a = \lambda\bar{H}\bar{H}_n + 4\mu H_1 H_0 > 0, \quad \bar{H} = H_1 + H_0, \quad \bar{H}_n = H_{n1} + H_{n0}$$

As a result, taking (5.6) into account, we have

$$[H_n] \geqslant -\mu[H]^3/(a H_1 H_0) > 0 \tag{5.11}$$

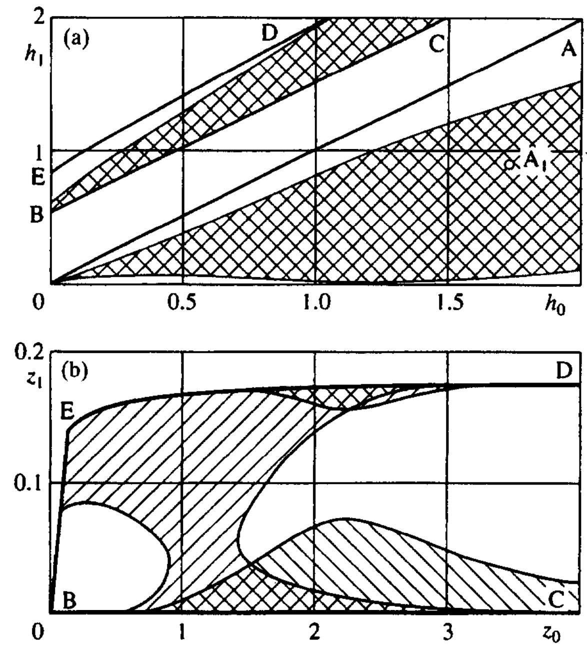

Fig. 1

This means that the total depth H_n must increase on passing through this discontinuity and, consequently, the domain of depths h_0, h_1 which are permitted by the Hugoniot conditions, subject to conditions (5.6) and (5.9), lies in the plane of the variables (h_0, h_1) above the line $h_1 = h_0 - [H]$.

We now transform inequality (5.7). To do this, using the formula $[\varphi] = [H_m^2] + \lambda\mu[h^2]$, we rewrite it as follows:

$$\tilde{h}([H_m^2]+\lambda\mu[h^2]) - 4\lambda\mu[h] = \tilde{h}\overline{H}_m[H_m] + \lambda\mu[h](\tilde{h}\overline{h} - 4) =$$
$$= (\tilde{h}\overline{H}_m[H_m] + \lambda\mu[h]^3)/(h_1 h_0) \leqslant 0, \quad \overline{H}_m = H_{m1} + H_{m0}$$

Hence, taking (5.9) into account, we obtain

$$[H_m] \leqslant -\lambda\mu[h]^3/(\tilde{h}\overline{H}_m) < 0 \tag{5.12}$$

This means that the total mass H_m per unit length of the flow decreases an passing through the discontinuity being considered and, consequently, the domain of depths h_0, h_1 which are permitted by the Hugoniot conditions, subject to conditions (5.6) and (5.9), lies in the plane of the variables (h_0, h_1) below the straight line $h_1 = h_0 - [H]/\lambda$.

Hence, in the case of (5.6) and (5.9) (that is, when the depth of the lower layer decreases and the depth of the upper layer increases on passing through the discontinuity) the depths of the upper layer which are permitted by the Hugoniot conditions satisfy the bilateral inequality

$$h_0 - [H]/\lambda > h_1 > h_0 - [H] \tag{5.13}$$

The set of these depths (obtained from the numerical solution of inequalities (5.7) and (5.10)) constitutes the band $BCDE$ in Fig. 1(a) for

$$H_0 = 1, \quad H_1 = 0,5, \quad \lambda = 0,5 \tag{5.14}$$

It follows from (5.13) that the width of this band tends to zero when $[H] \to 0$ or $\lambda \to 1$ and increases without limit when $\lambda \to 0$.

It is interesting to note that, if the laws of conservation of local momentum in the layers (4.2) are taken as the basis laws, instead of (4.4) and (4.7), the Hugoniot conditions corresponding to them will have the form

$$[v^2/2+g(H+h)]=\frac{q^2}{2}[h^{-2}]+g[H_n]=0$$

$$[V^2/2+g(H+\lambda h)]=\frac{Q^2}{2}[H^{-2}]+g[H_m]=0$$

and the discontinuities in the depths, which are permitted in this case, will thereby be determined by the inequalities

$$[H_n][h]\geqslant 0,\quad [H_m][H]\geqslant 0$$

which, as can be easily seen, are equivalent in the case of condition (5.6) to inequalities (5.8) and (5.13). This means that the set of discontinuities in the depth, permitted by the Hugoniot conditions, of the simplest incorrect system (4.1), (4.2) contains within itself the set of discontinuities in the depth, permitted by the Hugoniot conditions, of the correct system (4.1), (4.4), (4.7).

6. SHOCK WAVES PERMITTED BY THE CONDITION OF CONVEXITY OF THE TOTAL ENERGY

In order that the flow on both sides of the discontinuity should satisfy the condition of the convexity of the total energy (4.8), we require, in particular, that it guarantees its hyperbolic form. In this case, two fundamentally different situations are possible: when the fluid in both layers flows in a single direction through the discontinuity (we will call such discontinuities *shock waves with unidirectional flows* (USW)), and when the fluid flows in different directions through the discontinuity in the lower and upper layers (we will call such discontinuities *shock waves with flows in different directions* (DSW)).

In the case of USW, the following additional constraints on the values of the depths on the two sides of the discontinuity follow from (4.8) when (5.4) is taken into account

$$|\bar{Q}/H_i-\bar{q}/h_i|<\psi_i,\quad i=0,1 \tag{6.1}$$

where

$$\bar{Q}=((4\lambda\mu[h]-\tilde{h}[\varphi])/[H^{-1}])^{1/2},\quad \bar{q}=(-(4\mu[h]+\tilde{H}[\varphi])/[h^{-1}])^{1/2} \tag{6.2}$$

$$\psi_i=\frac{b_i}{\lambda r_i}(\bar{b}H_i(c_i-(c_i^2-4\lambda^2\mu r_i^3)^{1/2}))^{1/2} \tag{6.3}$$

$$b_i=1+\lambda r_i,\quad c_i=1+\lambda^2 r_i^3,\quad r_i=h_i/H_i$$

The depths which satisfy the convexity conditions (6.1) were determined numerically. They are shown in Fig. 1(a) in the form of the hatched sets, located within the domains (h_0, h_1) and $BCDE$, which are permitted by the Hugoniot conditions (5.1)–(5.3).

In the case of DSW, the convexity conditions (4.7) and (4.8) take the form

$$|\bar{Q}/H_i+\bar{q}/h_i|<\psi_i,\quad i=0,1 \tag{6.4}$$

Numerical calculations show that the constraints (6.4) are far stronger than (6.1) and the depths which are satisfied by them are located in Fig. 1(a) only inside the strip $BCDE$. Therefore, when

$$H_0=1,\quad H_1=0.75,\quad \lambda=0.5 \tag{6.5}$$

the sets of these depths are shown in Fig. 1(b) in the special system of coordinates z_0z_1, where

$$z_0=(h_1+h_0+[H])/\sqrt{2},\quad z_1=(h_1-h_0+[H])/\sqrt{2} \tag{6.6}$$

The z_0 axis of this system of coordinates lies on the line $h_1 = h_0 - [H]$ and the z_1 axis of the system is perpendicular. The depths (h_0, h_1), which satisfy conditions (6.4), are shown in Fig. 1(b) in the form of criss-cross hatched sets which are adjacent to the lower boundary BC and the upper boundary ED of the domain BCDE, shown in Fig. 1(b) in far greater detail than in Fig. 1(a). Here, in Fig. 1(b) (unlike in Fig. 1(a), not only the intersection of the sets (6.4) is shown but each of these sets is also shown separately: the first of them, which is obtained when $i = 0$, is shown hatched with straight lines sloping to the right while, the second, which is obtained when $i = 1$, is shown hatched with lines sloping to the left.

The numerical calculations showed that, in the case of a continuous change in the parameters H_1 and λ (which, in the case of Fig. 1(a) are determined by formulae (5.14) and, in the case of Fig. 1(b), by formulae (6.5)), the domains of convexity of the total energy also change continuously. However, the pattern of their arrangement remains practically unchanged.

7. STABLE SHOCK WAVES

In the case of total energy closing conservation law (4.6), the "entropy" condition for stability (2.4), when account is taken of the fact that $D = 0$, has the form

$$[FQ+\lambda fq]>0,\quad F=V^2+2gH_m,\quad f=v^2+2gH_n \tag{7.1}$$

Using the fact that, by virtue of (5.3), $[F] = [f]$, we obtain the inequality

$$\alpha[f]=\alpha[v^2+2gH_n]>0 \tag{7.2}$$

which is the two-layer analogue of the single-layer stability condition (3.9).

Let us assume that the total momentum (4.3) is positive. Then, using relations (5.4) and (6.2), inequality (7.2) can be rewritten in the following form

$$\begin{aligned}
&2\tilde{b}[f]/g=4\tilde{b}[H_n]+\bar{q}^2[h^{-2}]=4\tilde{b}[H_n]-\tilde{h}(4\mu[h]+\tilde{H}[\varphi])=\\
&=4(\tilde{h}+\lambda\tilde{H})[H_n]-4\mu\tilde{h}[h]-\tilde{h}\tilde{H}(\lambda[H_n^2]+\mu[H^2])=\lambda\tilde{H}[H_n](4-\tilde{h}\bar{h})+\tilde{h}[H_m](4-\tilde{H}\bar{H})=\\
&=(\lambda\bar{H}[-H_n][h]^2+\bar{h}[-H_m][H]^2)/(H_1H_0h_1h_0)>0
\end{aligned} \tag{7.3}$$

This inequality is directly satisfied in the case of conditions (5.6) and (5.8), which mean that the depth of the fluid in the two layers increases on passing through the discontinuity in the direction of the x axis (the set of such depths is located within the angle A_0h_0 in Fig. 1(a). In the case of conditions (5.6) and (5.9), inequality (7.3) selects a certain subset in the set of depths which are permitted by the Hugoniot conditions (5.1)–(5.4). In Fig. 2(a), this subset is indicated by hatching with lines sloping to the left, while the set of depths which satisfy the convexity conditions (6.1) is indicated by hatching with lines sloping to the right; as a result, the common part has criss-cross hatching.

Hence, the USW, for which $Q > 0$ and $q > 0$, are stable for depths with values which lie within the angle $A0h_0$ in Fig. 1(a) and within the domain in Fig. 2(a) indicated by the hatching with lines sloping to the left. At the same time, the USW, for which $Q > 0$ and $q < 0$, are only stable at depths, the values of which, in Fig. 2(a) fill that part of the domain $BCDE$ where there is no hatching with lines sloping to the left.

The profiles of the three qualitatively different stable discontinuous waves are shown in Fig. 3 which correspond to the point A_1 in Fig. 1(a) and to the points A_2 and A_3 in Fig. 2(a). the discontinuous wave corresponding to the point A_1 is shown in Fig. 3(a), the point A_2 in Fig. 3(b) and to the point A_3 in Fig. 3(c). In Fig. 3, the interfaces of the layers are depicted by the bold-face lines and their free surfaces by the thinner lines. The arrows denote the directions of the flows in the layers. The values of the flow rates and velocities in the layers presented in Fig. 3 were calculated assuming the acceleration due to gravity $g = 9.8$.

In the case of the DSW, for which $Q > 0$ and $q < 0$, the inequality $\alpha = Q + \lambda q > 0$ takes the form $\bar{Q} - \lambda\bar{q} > 0$, and the quantities $\bar{Q}$ and $\bar{q}$ are determined using formulae (6.2). The set of depths (h_0, h_1) which satisfy this condition is shown in Fig. 2(b) in the form of the two domains indicated by the hatching with lines sloping to the left. Similarly, the DSW, for which $Q < 0$ and $q > 0$ will be stable when the inequality $(\bar{Q}-\lambda\bar{q})[f] < 0$ is satisfied and, consequently, the values of the depths which are permissible for them occupy the unhatched parts of the domain $BCDE$ in Fig. 2.

The depths (h_0, h_1), which satisfy the convexity conditions (6.4), are shown in Fig. 2(b) (in the same way as in Fig. 1(b) in the form of the two criss-cross hatched sets. It follows from Fig. 2(b) that, in the case of conditions (6.5), the DSW which satisfy the convexity condition (6.4) are stable only when the flow is directed along the x axis in the lower layer and in the opposite direction in the upper layer, that

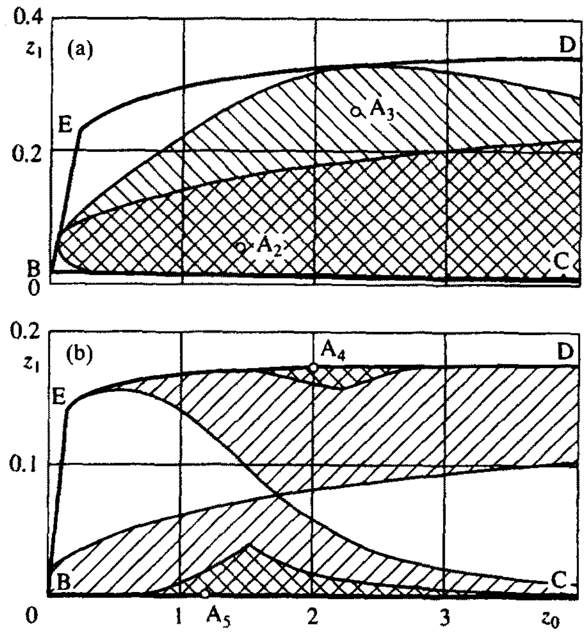
0.4
z1
(a)
D
E
A3
0.2
A2
B
C
0
0.2
z1
(b)
A4
D
E
0.1
B
C
A5
0
1
2
3
z0

Fig. 2

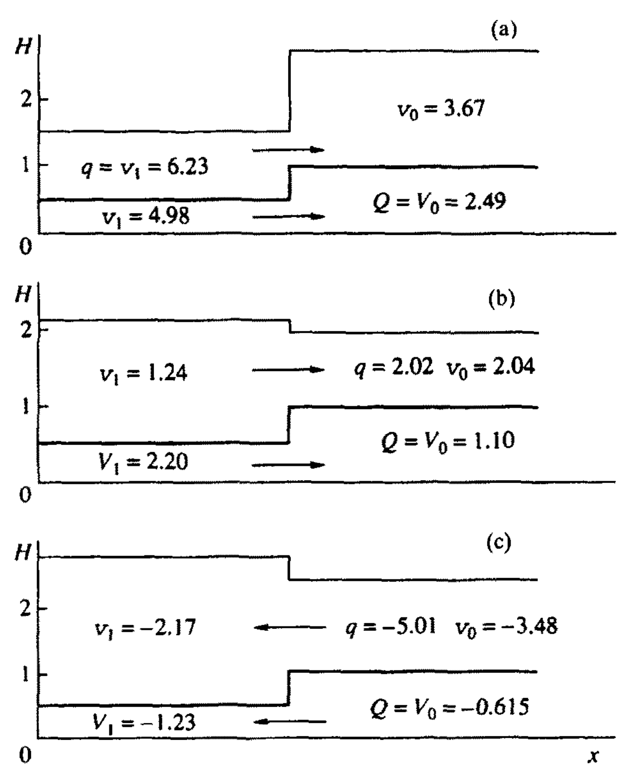
(a)
H
2
1
0
v0 = 3.67
q = v1 = 6.23
Q = V0 = 2.49
v1 = 4.98
(b)
v1 = 1.24
q = 2.02 v0 = 2.04
Q = V0 = 1.10
V1 = 2.20
(c)
v1 = −2.17
q = −5.01 v0 = −3.48
Q = V0 = −0.615
V1 = −1.23
x

Fig. 3

is, when $Q > 0$ and $q < 0$. Numerical calculations showed that this result also remains true in the case of other permissible values of the parameters H_1 and λ.

8. "NATURAL DAMS"

The shock waves for which there is no fluid flow through the discontinuity in one of the layers are intermediate between USW's and DSW's. Such unusual two-layer discontinuous waves are called natural dams in the lower layer (when $Q = 0$) and upper layer (when $q = 0$). It can be shown that, when $Q = 0$, the permissible depths of the upper layer lie on the upper boundary DE of the domain $BCDE$ and, when $q = 0$, they lie on its lower boundary BC (compare Figs 1(b) and 2(b).

When $Q = 0$, it follows from the first equation of (5.4) that

$$4\lambda\mu[h] = \tilde{h}[\varphi] \tag{8.1}$$

Expressing $[\varphi]$ from here and substituting the result into the second equation of (5.4), we find that the flow rate in the upper layer is given by the formula

$$q^2 = 2\mu g h_1^2 h_0^2 / \bar{h} \tag{8.2}$$

For such natural dams, the convexity conditions (6.1) take the form

$$2h_1(\mu\tilde{b}/\bar{h})^{1/2} < \psi_0, \quad 2h_0(\mu\tilde{b}/\bar{h})^{1/2} < \psi_1 \tag{8.3}$$

Here, the functions ψ_i are given by (6.3) and the parameter λ, taking account of the fact that $\mu = 1 - \lambda$ and $\varphi = H^2 + \lambda h^2 + 2\lambda Hh$, is found from Eq. (8.1) using the formula

$$\lambda = \left((\beta^2 - 16\tilde{h}[h][H^2])^{1/2} - \beta\right)/(8[h]), \quad \beta = \tilde{h}[h^2 + 2Hh] - 4[h] \tag{8.4}$$

As a result of constraint (8.3), unlike in (6.1) when $H_0 = 1$, they depend solely on the three arbitrary parameters H_1, h_0 and h_1.

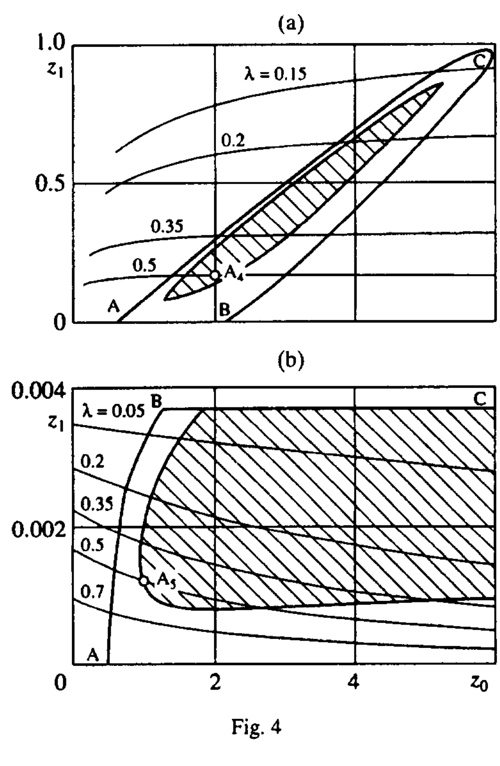

Fig. 4

The set of depths (h_0, h_1), which satisfy the convexity condition (8.3) when $H_0 = 1$ and $H_1 = 0.75$ is plotted in Fig. 4(a) in the system of coordinates (6.6) in the form of the domain which is bounded by the closed curve ABC. The isolines of λ, calculated using formula (8.4), are shown by the thin curves in this figure. Numerical calculations showed that, when H_1 increases up to $H_0 = 1$, the domain ABC is "stretched out" in an unbounded manner and, when H_1 is reduced, it shrinks. Here, a critical value $H_1^* \approx 0.41H_0$ exists such that, when $H_1 < H_1^*$, the domain ABC ceases to exist, that is, natural dams in the lower layer are not permitted by the convexity conditions (8.3) for such values of the depths H_1.

From the energy condition of stability (7.2) when $Q = 0$ and (8.2) is taken into account, we obtain

$$q[v^2 + 2gH_n] = q(q^2[h^{-2}] + 2g[H_n]) = 2gq(\mu h_1^2 h_0^2[h^{-2}]/\bar{h} + [H_n]) =$$
$$= 2gq([H_n] - \mu[h]) = 2gq[H_m] > 0$$

It follows, as a result when (5.12) is taken into account, that, in order for a natural dam to be stable in the lower layer, it is necessary for the inequality $q < 0$ to be satisfied, which means that the flow in the upper layer is in the opposite direction to the x axis. An example of such a natural dam (which corresponds to the point A_4 in Fig. 2(b) and Fig. 4(a) is shown in Fig. 5(a).

We will now consider a natural dam in the upper layer for which $q = 0$. In this case, it follows from the second equation of (5.4) that

$$4\mu[h] + \tilde{H}[\varphi] = 0 \tag{8.5}$$

Expressing $[\varphi]$ using this equation and substituting the result into the first equation of (5.4), we obtain that the flow rate in the lower layer is given by the formula

$$Q^2 = 2\mu g[h]/([H^{-1}]\tilde{H}) \tag{8.6}$$

In the case of such natural dams, the convexity conditions (6.1) take the form

$$2H_1(-\mu[h]\bar{b}/[H^2])^{1/2} < \psi_0, \quad 2H_0(-\mu[h]\bar{b}/[H^2])^{1/2} < \psi_1 \tag{8.7}$$

where the functions ψ_i are found from formula (6.3) and the parameter λ is found from Eq. (8.5) using the formula

$$\lambda = (4[h] - \tilde{H}[H^2])/(\tilde{H}[h^2 + 2Hh] + 4[h]) \tag{8.8}$$

In the system of coordinates (6.6), the set of depths (h_0, h_1), which satisfy the convexity conditions (8.7) when $H_0 = 1$ and $H_1 = 0.75$, is the domain bounded by the curve ABC and the z_0 axis in Fig. 4(b).

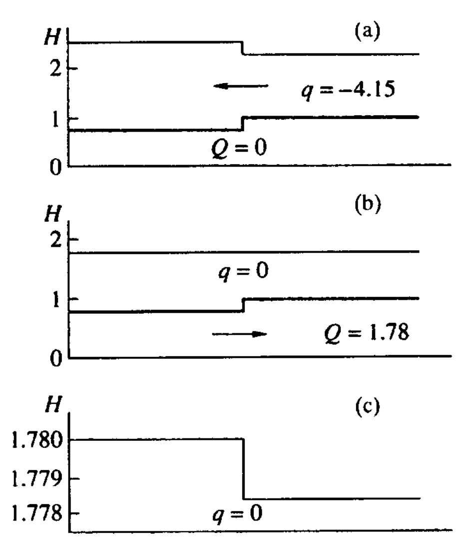

Fig. 5

The isolines of λ, calculated using formula (8.8), are shown by the thin curves. It follows from this figure (and, also, from the results of other calculations) that the distinctive feature of a natural dam in the upper layer is that, for finite H_1, they can only exist when there is a very small jump in the total depth on the line of the discontinuity ($[H_n] < 0.004$) in the case of Fig. 4(b). This means that the upper boundary of the flow in the case of such discontinuities will be visually perceived as a horizontal surface (see Fig. 5(b).

When $q = 0$, we obtain from the energy condition for stability (7.2)

$$Q[V^2 + 2gH_m] = Q(Q^2[H^{-2}] + 2g[H_m]) = 2gQ([H_m] + \mu[h][H^{-2}]/([H^{-1}]\tilde{H})) =$$
$$= 2gQ([H_m] + \mu[h]) = 2gQ[H_n] > 0$$

Taking account of (5.11), it follows from this inequality that, in order for a natural dam to be stable in the upper layer, it is necessary that $Q > 0$, that is, the flow in the lower layer must be directed along the x axis. An example of such a natural dam (which corresponds to the point A_5 in Fig. 2(a) and Fig. 4(b) is shown in Fig. 5(b) and 5(c). In Fig. 5(c), the discontinuity in the free surface, corresponding to this natural dam, is shown in a smaller scale along the H axis.

9. TWO-LAYER "SHALLOW WATER" IN THE TWO-DIMENSIONAL CASE

The possibility of its natural extension to the spatially two-dimensional case is a further important and simultaneously independent criterion of the correctness of the proposed complete system of conservation laws for the model of two-layer "shallow water". Taking this into account, we will now consider the two-dimensional flow of two-layer "shallow water" over a horizontal bed assuming that there is no friction between the water and the bed and between the layers. Such a flow is determined by the depths h and H and the vertically averaged horizontal velocities $\mathbf{v} = (\upsilon_1, \upsilon_2)$, $\mathbf{V} = (V_1, V_2)$, of the upper and lower layers respectively, which depend on the time t and the horizontal coordinates x and y.

The differential equations describing this flow can be obtained from the integral laws of conservation of mass and total momentum in each of the fluid layers using a method similar to that proposed in [2] for deriving the equations of single-layer "shallow water" in the one-dimensional case. The above-mentioned equations have the form

$$h_t + \operatorname{div}\mathbf{q} = 0, \quad H_t + \operatorname{div}\mathbf{Q} = 0 \tag{9.1}$$

$$\mathbf{q}_t + \mathbf{div}(\mathbf{q} \otimes \mathbf{v}) + gh\nabla H_n = 0, \quad \mathbf{Q}_t + \mathbf{div}(\mathbf{Q} \otimes \mathbf{V}) + gH\nabla H_m = 0 \tag{9.2}$$
$$\mathbf{q} = (q_1, q_2) = h\mathbf{v}, \quad \mathbf{Q} = (Q_1, Q_2) = H\mathbf{V}, \quad \operatorname{div}\mathbf{q} = q_{1x} + q_{2y}$$
$$\mathbf{div}(\mathbf{q} \otimes \mathbf{v}) = ((q_1 \upsilon_1)_x + (q_1 \upsilon_2)_y, \ (q_2 \upsilon_1)_x + (q_2 q_2)_y)$$

Equations (9.1) are the laws of conservation of mass in the layers and Eqs (9.2) describe the change in the total momentum in each of the layers.

Euler's equations

$$\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} + g\nabla H_n = 0, \quad \mathbf{V}_t + (\mathbf{V}\nabla)\mathbf{V} + g\nabla H_m = 0 \tag{9.3}$$

are the differential corollaries of system (9.1), (9.2), which describe the change in the local momentum in the layers, the law of conservation of total momentum

$$\boldsymbol{\alpha} = \mathbf{Q} + \lambda\mathbf{q} \tag{9.4}$$

in the form

$$\boldsymbol{\alpha}_t + \mathbf{div}(\mathbf{Q} \otimes \mathbf{V}) + \lambda\,\mathbf{div}(\mathbf{q} \otimes \mathbf{v}) + g\nabla\varphi/2 = 0 \tag{9.5}$$

and the law of conservation of total energy

$$e = \mathbf{Q} \cdot \mathbf{V} + \lambda\mathbf{q} \cdot \mathbf{v} + g\varphi \tag{9.6}$$

in the form

$$e_t + \operatorname{div}(\bar{F}\mathbf{Q} + \lambda\bar{f}\mathbf{q}) = 0 \tag{9.7}$$

where

$$\bar{F} = |\mathbf{V}|^2 + 2gH_m, \quad \bar{f} = |\mathbf{v}|^2 + 2gH_n \tag{9.8}$$

On transforming equations (9.3) using the formula

$$(\mathbf{v}\nabla)\mathbf{v} = \nabla(|\mathbf{v}|^2/2) - [\mathbf{v} \times \mathbf{rot}\,\mathbf{v}] = \nabla(|\mathbf{v}|^2/2) - w\mathbf{v}^{\perp}$$

in which $w = \upsilon_{2x} - \upsilon_{1y}$ is a plane vortex and $\mathbf{v}^{\perp} = (\upsilon_2 - \upsilon_1)$ is a vector which is perpendicular to the velocity vector $\mathbf{v}$, we obtain the equations

$$\mathbf{v}_t + \nabla\bar{f}/2 = w\mathbf{v}^{\perp}, \quad \mathbf{V}_t + \nabla\bar{F}/2 = W\mathbf{V}^{\perp} \tag{9.9}$$

which are the two-dimensional analogue of the divergent equations (4.2). Equations (9.9) themselves are only divergent in the case of potential flows in the layers for which $w = W = 0$. In this case, system (9.1), (9.9) can be obtained using the potential method in a similar way to that employed in [1] when deriving the one-dimensional equations (4.1) and (4.2).

On-applying the curl operator to Eqs (9.9), we obtain the scalar divergent equations

$$w_t + \operatorname{div}(w\mathbf{v}) = 0, \quad W_t + \operatorname{div}(W\mathbf{V}) = 0 \tag{9.10}$$

which are the laws of conservation of plane vortices w and W in each of the layers of a two-layer fluid. In the case of smooth flows, these conservation laws are, in essence, the spatially two-dimensional analogue of Thompson's theorem on the conservation of circulation of velocity.

By analogy with the one-dimensional case, we take the scalar laws of conservation of mass in the layers (9.1), the vector law of conservation of total momentum (9.5) and and the vector equation

$$\boldsymbol{\gamma}_t + (\mathbf{V}\nabla)\mathbf{V} - (\mathbf{v}\nabla)\mathbf{v} - \mu g\nabla h = 0 \tag{9.11}$$

which follows from (9.3) and describes the change in the velocity jump $\boldsymbol{\gamma} = \mathbf{V} - \mathbf{v}$ at the interface of the layers, as the basic conservation laws of system (9.1)–(9.10). Next, it will be shown that, despite the non-divergence of Eq. (9.11), it can be used to obtain the Hugoniot conditions correctly on the lines of discontinuities.

As the closing conservation law, we take the law of conservation of total energy (9.7). Here, as in the one-dimensional case, the total energy (9.6), written as the function

$$e(h, H, \boldsymbol{\alpha}, \boldsymbol{\gamma}) = (|\boldsymbol{\alpha}|^2 + \lambda hH|\boldsymbol{\gamma}|^2)/(H + \lambda h) + g(H^2 + \lambda h^2 + 2\lambda hH) \tag{9.12}$$

which depends on the basic variables $\mathbf{u} = (h, H, \alpha, \gamma)$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$, is convex in the case of conditions (4.8) and (4.9) and, in this function, the scalar quantity γ has to be replaced by the plane vector γ. This convexity condition (which, in the case of two-dimensional flows, is a natural alternative requirement of hyperbolicity) will later be used to select the permissible two-layer flows.

10. THE HUGONIOT CONDITIONS IN THE TWO-DIMENSIONAL CASE

In order to obtain the Hugoniot conditions, which the system of basic conservation laws (9.1), (9.5), (9.11) permits, we consider any element of a smooth line of discontinuity of a certain generalized solution of it and use a system of coordinates associated with this element with the x axis directed along the normal to it and the y axis directed along the tangent to it. We shall denote the normal and tangential components of vector quantities $\mathbf{f}$ by f_n and f_τ and, here, for convenience in making comparisons with the formulae for the one-dimensional case, we shall sometimes omit the subscript n on the normal components. We shall also take into account that the velocity of a line of discontinuity $D = 0$ in the system of coordinates considered and all the quantities are fairly continuous functions of the variable y.

As a result, as in the one-dimensional case, from the scalar equations (9.1) and the normal components of vector equations (9.5) and (9.11) we obtain relations (5.1)–(5.3), from which formulae (5.4) and the constraints on the jumps in the depths on at discontinuity (5.5) follow. Hence, the analysis carried out above of the jumps in the depths at the discontinuity, which are permitted by the Hugoniot conditions, can be transferred as a whole to the two-dimensional case.

Taking account of (5.1), from the tangential component of the law of conservation of total momentum (9.5), we obtain the relation

$$[QV_\tau + \lambda q v_\tau] = Q[V_\tau] + \lambda q[v_\tau] = 0 \tag{10.1}$$

which establishes the relation at the discontinuity between the normal components of the flow rates in the layers and the jumps in the tangential components of the velocity. In this connection we note that, in the case of the single-layer, two-dimensional model of "shallow water", for which the law of conservation of total momentum has the form

$$\mathbf{q}_t + \mathbf{div}\,(\mathbf{q} \otimes \mathbf{v}) + g\nabla h^2/2 = 0$$

the Hugoniot condition for its tangential component, like (10.1), can be written in the following manner: $q[\upsilon_\tau] = 0$. As a result, it follows that the single-layer two-dimensional model, which is similar to the equations of classical hydrodynamics [16], only allows of two types of discontinuities: shock waves for which $q = q_n \neq 0$, $[\upsilon_\tau] = 0$ and contact discontinuities for which $q = q_n = 0$, $[\upsilon_\tau] \neq 0$. At the same time, it follows from (10.1) that the two-layer, two-dimensional model not only allows of shock waves and contact discontinuities but, also, discontinuities of a mixed type, for which both the fluid flow through the discontinuity as well as the jump in its tangential velocity component are non-zero.

Since the tangential component of Eq. (9.11) cannot be written in divergent form with respect to the variable x, it cannot be used directly to obtain a further relation at the discontinuity. In order to do this, is necessary to derive a scalar integral corollary of Eq. (9.11) which, taking (9.9) into account, is conveniently written in the form

$$\gamma_t + \nabla(\bar{F} - \bar{f})/2 = W\mathbf{V}^\perp - w\mathbf{v}^\perp \tag{10.2}$$

Introducing the notation

$$\varphi = (\bar{F} - \bar{f})/2, \quad \psi = W\mathbf{V}^\perp - w\mathbf{v}^\perp$$

we write the integral analogue of conservation law (10.2) as

$$\int (\gamma g_t + \varphi\nabla g - \psi g)\,dtdxdy = 0, \quad \forall g(t,x,y) \in C_0^\infty \tag{10.3}$$

where C_0^∞ is a class of infinitely differentiable, finite functions. In formula (10.3) and henceforth, if the domain of integration is not shown, it means that integration is carried out with respect to the carrier of the function g.

To obtain the required integral corollary of conservation law (10.3), we use the method developed in [17]. Substituting the first derivative of the trial function, instead of the trial function g itself, into the first component of vector equation (10.3) and substituting the derivative g_x instead of g into the second component of this equation, we obtain the following integral corollaries of conservation law (10.3)

$$\int (\gamma_1 g_{yt} + \varphi g_{yx} - \psi_1 g_y)\,dtdxdy = 0, \quad \forall g \in C_0^\infty \tag{10.4}$$

$$\int (\gamma_2 g_{xt} + \varphi g_{xy} - \psi_2 g_x)\,dtdxdy = 0, \quad \forall g \in C_0^\infty \tag{10.5}$$

Integrating the first term of these equations by parts and then subtracting Eq. (10.5) from Eq. (10.4), we obtain the scalar conservation law

$$\int ((W - w)g_t + (WV_1 - wv_1)g_x + (WV_2 - wv_2)g_y)\,dtdxdy = 0, \quad \forall g \in C_0^\infty$$

which, when written in differential form, becomes

$$(W - w)_t + \mathrm{div}\,(W\mathbf{V} - w\mathbf{v}) = 0 \tag{10.6}$$

Hence it follows that, in the system of coordinates associated with the line of discontinuity, the additional Hugoniot condition

$$[V_n W - \upsilon_n w] = 0 \tag{10.7}$$

holds, which signifies continuity on the line of the discontinuity of the jump in the vorticity at the interface of the layers. In this case, each of the vortices separately will not be conserved at the discontinuity in the general case, despite the fact that the change in the vortices in smooth flows is described by divergent equations (9.10).

Hence, in the case of non-potential flows, a vector law of conservation of the jump in velocity (9.11) is realized at discontinuties in the form of a law of conservation of the jump in the normal component of the velocity (5.3) and a law of conservation of the jump in the vorticity (10.6), (10.7).

11. THE CONDITIONS OF CONVEXITY OF THE TOTAL ENERGY IN THE TWO-DIMENSIONAL CASE

The requirement that the total energy (9.6) should be convex (4.8) on both sides of a line of discontinuity leads, first of all, to the following constraints on the jump in the tangential component of the velocity at the interface of the layers

$$|\gamma_{\tau i}| = |V_{\tau i} - \upsilon_{\tau i}| < (g/(2\bar{b}))^{1/2}\psi_i, \quad i = 0, 1 \tag{11.1}$$

where the functions ψ_i are determined using formula (6.3). When conditions (11.1) are satisfied and (5.4) is taken into account, the constraints on the jumps in the depths at the discontinuity, which are additional to (5.5), follow:

for USW

$$|\bar{Q}/H_i - \bar{q}/h_i| < (\psi_i^2 - 2\bar{b}\gamma_{\tau i}^2/g)^{1/2}, \quad i = 0, 1 \tag{11.2}$$

and for DSW

$$|\bar{Q}/H_i + \bar{q}/h_i| < (\psi_i^2 - 2\bar{b}\gamma_{\tau i}^2/g)^{1/2}, \quad i = 0, 1 \tag{11.3}$$

where the quantities $\bar{Q}$ and $\bar{q}$ are found from formula (6.2).

If there is no jump in the tangential component of the velocity on both sides of the discontinuity at the interface of the layers, that is, if $\upsilon_{\tau i} = V_{\tau i}$, then constraints (11.2) and (11.3) are identical to the analogous constraints (6.1) and (6.4) for one-dimensional flows. If, however, $V_{\tau i} \neq \upsilon_{\tau i}$, then conditions (11.2) and (11.3) become much more rigorous. In this case, as follows from (10.1), the jumps in the tangential components of the velocities in the layers $[V_\tau]$ and $[\upsilon_\tau]$ have different signs in the case of USW and the same signs in the case of DSW.

The sets of depths, which satisfy convexity conditions (11.2) and (11.3), were determined numerically. For

$$H_0 = 1, \quad H_1 = 0.5, \quad \lambda = 0.5 \tag{11.4}$$

$$V_{\tau 0} = 0, \quad \upsilon_{\tau 0} = 1, \quad V_{\tau 1} = 1 \Rightarrow [V_\tau] = 1, \quad \gamma_{\tau 0} = -1 \tag{11.5}$$

the set of depths (h_0, h_1), which satisfy inequality (11.2), is shown in Fig. 6(a) in the form of the two criss-cross domains located within the set $A0h_0$ and $BCDE$, which are permitted by the Hugoniot conditions (5.4) (here, the quantity $\upsilon_{\tau 1}$, occurring in (11.2) when $i = 1$, was determined from the Hugoniot condition (10.1)). These domains are also located within the two sets in Fig. 6(a) which are shown hatched with lines sloping to the right are obtained from inequalities (1.2) when $\gamma_{\tau i} = 0$, that is, when there is no jump on both sides of the line of discontinuity in the tangential component of the velocity at the interface of the layers (these sets are shown separately in Fig. 1(a)).

Constraints (11.3) are far stronger than (11.2) and, as was shown above (see Fig. 1(b)), even when $\gamma_{\tau i} = 0$, the depths which satisfy inequality (11.3) are only located in Fig. 6(a) within the strip $BCDE$. Hence, when

$$H_0 = 1, \quad H_1 = 0.75, \quad \lambda = 0.5 \tag{11.6}$$

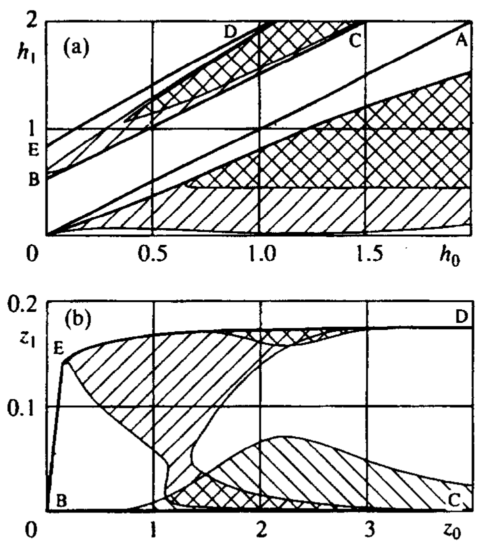

Fig. 6

$$V_{\tau 0} = 0, \quad \upsilon_{\tau 0} = 0.25, \quad V_{\tau 1} = 0.25 \Rightarrow [V_\tau] = 0.25, \quad \gamma_{\tau 0} = -0.25 \tag{11.7}$$

the depths (h_0, h_1) which satisfy conditions (11.3) are shown in Fig. 6(b) in the system of coordinates (6.6). The depths fill out the two criss-cross sets adjacent to the lower boundary BC and the upper boundary ED of the domain $BCDE$. Here in Fig. 6(b) (as in Fig. 1(b)) not only the intersection of the sets (11.3) is shown but also each of these sets separately: the first of them, which is obtained when $i = 0$, is shown hatched with lines sloping to the right while the second, obtained for $i = 1$, is shown hatched with lines sloping to the left.

12. STABLE DISCONTINUITIES IN THE TWO-DIMENSIONAL CASE

It follows from the law of conservation of total energy (9.7) that the energy condition of stability, which assumes that there is a loss in the total energy at a discontinuity, has the following form

$$[\bar{F}Q_n + \lambda \bar{f} q_n] = Q_n[F + V_\tau^2] + \lambda q_n [f + \upsilon_\tau^2] > 0 \tag{12.1}$$

$$F = V_n^2 + 2gH_m, \quad f = \upsilon_n^2 + 2gH_n$$

Since it follows from the Hugoniot conditions (5.3) and (10.1) that

$$[F] = [f], \quad Q_n[V_\tau] = -\lambda q_n [\upsilon_\tau] \tag{12.2}$$

on transforming, taking account of inequality (12.1), we obtain

$$\alpha_n [F] + Q_n \bar{\gamma}_\tau [V_\tau] > 0 \tag{12.3}$$

$$\alpha_n = Q_n + \lambda q_n, \quad \bar{\gamma}_\tau = \gamma_{\tau 1} + \gamma_{\tau 0} = V_{\tau 1} + V_{\tau 0} - \upsilon_{\tau 1} - \upsilon_{\tau 0}$$

As a result, in the case of USW directed along the x axis ($Q_n > 0$, $q_n > 0$), using the notation (6.2), we have

$$F_1 = (\bar{Q} + \lambda \bar{q}) f_1 + \bar{Q} f_2 > 0, \tag{12.4}$$

$$f_1 = \bar{Q}^2 [H^{-2}] + 4\bar{b}[H_m], \quad f_2 = 2\bar{b}\bar{\gamma}_\tau [V_\tau]/g$$

In the case of USW directed in the opposite direction to the x axis ($Q_n < 0, q_n < 0$), the energy inequality (1.23) takes the form $F_1 < 0$.

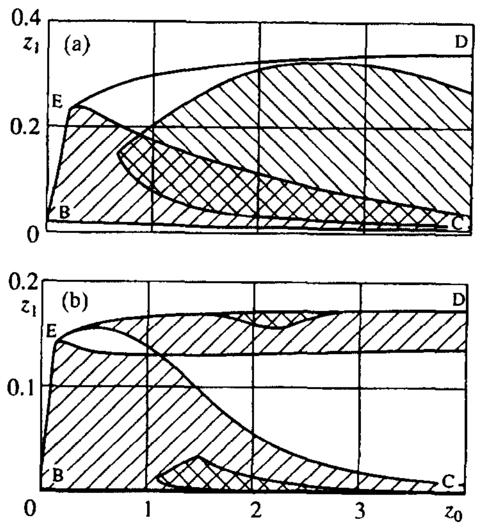

Fig. 7

Numerical calculations showed that, in the case of the parameters (11.4), (11.5), inequality (12.4) is satisfied in the whole of the domain A_0h_0 and in a certain subdomain of $BCDE$ in Fig. 6(a). The above-mentioned subdomain is shown in Fig. 7(a) in the system of coordinates (6.6) by the hatched area with lines sloping to the left, while the set of depths which satisfy convexity conditions (11.2) are shown by the hatching with lines sloping to the right: as a result, their common domain is shown cross-hatched. A comparison of this figure with Fig. 2(a) shows that, in this case, the existence of a jump in the tangential component of the velocity leads to a considerable contraction of that part of the domain $BCDE$ within which discontinuities with a positive direction of the flows in the layers ($Q_n > 0, q_n > 0$) are stable and, correspondingly, to a noticeable expansion of the part of this domain within which discontinuities with a negative direction of the flows in the layers ($Q_n < 0, q_n < 0$) are stable.

In the case of DSW, directed in the lower layer along the x axis and, in the upper layer, in the opposite direction, that is, when $Q_n > 0$ and $q_n < 0$, we obtain from (2.3)

$$F_2 = (\bar{Q} - \lambda\bar{q})f_1 + \bar{Q}f_2 > 0 \tag{12.5}$$

The set of depths (h_0, h_1), which, for conditions (11.6) and (11.7), satisfy inequality (12.5), is shown in Fig. 7(b) in the form of the two domains which have been hatched with lines sloping to the left. In the case of DSW, directed along the x axis in the upper layer and in the opposite direction in the lower layer, that is, when $Q_n < 0$ and $q_n > 0$, the energy inequality (12.5) changes sign and takes the form $F_2 < 0$. This means that the depths permitted by these discontinuities fill out the unhatched part of the domain $BCDE$ in Fig. 7(b).

The depths (h_0, h_1) which satisfy the convexity conditions (11.3) are shown in Fig. 7(b) in the form of two criss-cross sets. It follows from Fig. 7(b) that, subject to conditions (11.6) and (11.7), the DSW which satisfy convexity condition (11.3) are only stable when the flow in the lower layer is along the x axis and the flow in the upper layer is in the opposite direction, that is, when $Q_n > 0$ and $q_n < 0$. Numerical calculations showed that this result also remains true in the case of other permissible values of the flow parameters.

13. STABLE CONTACT DISCONTINUITIES IN THE LAYERS

Intermediate between two-dimensional USW ($Q_nq_n > 0$) and DSW ($Q_nq_n < 0$) are those two-layer discontinuities for which there is no fluid flow through the discontinuity in one of the layers, that is $Q_n = 0$ or $q_n = 0$. In this case, taking (10.1) into account, three different situations are possible:

(a) $Q_n = q_n = 0$. In this case, there is a contact discontinuity in both layers. We shall refer to such a two-layer discontinuity as a full contact discontinuity;

(b) $Q_n = 0, q_n \neq 0, [\upsilon_\tau] = 0$. In this case, there is a contact discontinuity in the lower layer and a shock wave in the upper layer. We shall refer to such a two-layer combined discontinuity as a contact discontinuity in the lower layer (CDLL);

(c) $q_n = 0, Q_n \neq 0, [V_\tau] = 0$. In this case, there is a shock wave in the lower layer and a contact discontinuity in the upper layer. We shall refer to such a two-layer discontinuity as a contact discontinuity in the upper layer (CDUL).

Note that the contact discontinuities in the layers become natural dams when $\gamma_{\tau i} = 0$ $(i = 0, 1)$, that is, when $V_{\tau 0} = V_{\tau 1}, \upsilon_{\tau 0} = \upsilon_{\tau 1}$, which have been studied in Section 8.

It follows from (12.3) that, when $Q_n = q_n = 0$, the total energy is conserved on passing through a discontinuity, which can be interpreted as an energy instability of a full contact discontinuity (the analogy with classical hydrodynamics shows itself here [16]). At the same time, the energy condition of stability (12.3) allows of contact discontinuities in the layers: for a CDLL, it takes the from $q_n[f] > 0$ and, in the case of a CDUL, the following form: $Q_n[F] > 0$. As was shown in Section 8, it follows from the inequality $q_n[f] > 0$ in the case of the conditions $H_0 > H_1$ that $q_n < 0$, and it follows from the inequality $Q_n[F] > 0$ that $Q_n > 0$. This means that, when $H_0 > H_1$, in order for a CDLL to be stable, it is necessary for the flow in the upper layer to be in the opposite direction to the x axis and, conversely, in order for a CDUL to be stable, it is necessary for the flow in the lower layer to be along the x axis.

In the case of a CDLL, it may be assumed without loss of generality that $\upsilon_{\tau 0} = \upsilon_{\tau 1} = 0$. Convexity conditions (11.2) and (11.3), when (8.1) and (8.2) are taken into account, then take the form

$$2\mu g h_1^2 / \bar{h} < g\psi_0^2/(2\tilde{b}) - V_{\tau 0}^2, \quad 2\mu g h_0^2 / \bar{h} < g\psi_1^2/(2\tilde{b}) - V_{\tau 1}^2 \tag{13.1}$$

where the functions ψ_i are determined from formula (6.3) and the parameter λ is found from Eq. (8.1).

As an example, the set of depths (h_0, h_1) which satisfy convexity conditions (13.1) when $H_0 = 1, H_1 = 0.75$ and when

$$\upsilon_{\tau 0} = \upsilon_{\tau 1} = 0, \quad V_{\tau 0} = 1, \quad V_{\tau 1} = 2 \Rightarrow [V_\tau] = 1, \quad \gamma_{\tau 0} = 1, \quad \gamma_{\tau 1} = 2$$

is plotted in Fig. 4(a) in the system of coordinates (6.6) in the form of a hatched domain. This domain is located within the set bounded by the closed curve ABC and is obtained from inequalities (13.1) when $V_{\tau 0} = V_{\tau 1} = 0$.

In the case of a CDUL, it can be assumed, without loss of generality, that $V_{\tau 0} = V_{\tau 1} = 0$. Then, convexity conditions (11.2) and (11.3), taking (8.5) and (8.6) into account, can be written in the form

$$2\mu g H_1^2[h]/[H^2] > \upsilon_{\tau 0}^2 - g\psi_0^2/(2\tilde{b}), \quad 2\mu g H_0^2[h]/[H^2] > \upsilon_{\tau 1}^2 - g\psi_1^2/(2\tilde{b}) \tag{13.2}$$

where the functions ψ_i are determined from formula (6.3) and the parameter λ is found from Eq. (8.5).

As an example, the set of depths (h_0, h_1), which satisfy convexity conditions (13.2) when $H_0 = 1$, $H_1 = 0.75$ and when

$$V_{\tau 0} = V_{\tau 1} = 0, \quad \upsilon_{\tau 0} = 1, \quad \upsilon_{\tau 1} = 2 \Rightarrow [\upsilon_\tau] = 1, \quad \gamma_{\tau 0} = -1, \quad \gamma_{\tau 1} = -2$$

are shown in Fig. 4(b) in the system of coordinates (6.6) in the form of a hatched domain. This domain is located within the set which is bounded by the curve ABC and the z_0 axis and is obtained from inequalities (13.4) when $\upsilon_{\tau 0} = \upsilon_{\tau 1} = 0$.

14. CONCLUSION

The above analysis of stable, discontinuous, two-layer waves is of a purely theoretical nature, and the possibility of the existence of such waves requires a special experimental investigation. Here, it is necessary to take into account that the discontinutities considered in this model are, in practice, whole transitional domains of wave-like flow of a fluid which can make the interpretation of the results obtained much more difficult.

A further, more detailed study of the special features of discontinuous flows of two-layer "shallow water" by direct numerical simulation of the complete system of conservation laws proposed in this paper is also of interest.

I wish to thank E. V. Shugrin for help in designing the graphical material.

The results in this paper were presented at the All-Russian Conference on "Mathematical Problems of Mechanics", coinciding with Academician L. V. Ovsyannikov's eightieth birthday year (19–22 April 1999, Novosibirsk).

This research was supported financially by a grant from the Committee for the Support of the Leading Scientific Schools (00–15–96–162).

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Translated by E.L.S.